

Gravitational instantons admit hyper-Kähler structure

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Abstract

We construct the explicit form of three almost complex structures that a Riemannian manifold with self-dual curvature admits and show that their Nijenhuis tensors vanish so that they are integrable. This proves that gravitational instantons with self-dual curvature admit hyper-Kähler structure. In order to arrive at the three vector valued 1-forms defining almost complex structure, we give a spinor description of real 4-dimensional Riemannian manifolds with Euclidean signature in terms of two independent sets of 2-component spinors. This is a version of the original Newman-Penrose formalism that is appropriate to the discussion of the mathematical, as well as physical properties of gravitational instantons. We shall build on the work of Goldblatt who first developed an NP formalism for gravitational instantons but we shall adopt it to differential forms in the NP basis to make the formalism much more compact. We shall show that the spin coefficients, connection 1-form, curvature 2-form, Ricci and Bianchi identities, as well as the Maxwell equations naturally split up into their self-dual and anti-self-dual parts corresponding to the two independent spin frames. We shall give the complex dyad as well as the spinor formulation of the almost complex structures and show that they reappear under the guise of a triad basis for the Petrov classification of gravitational instantons. Completing the work of Salamon on hyper-Kähler structure, we show that the vanishing of the Nijenhuis tensor for all three almost complex structures depends on the choice of a self-dual gauge for the connection which is guaranteed by virtue of the fact that the curvature 2-form is self-dual for gravitational instantons.

1 Introduction

Gravitational instantons are described by 4-dimensional real Riemannian metrics with self-dual curvature. We shall prove that they admit rich complex structure, namely hyper-Kähler structure. For a general metric with self-dual curvature 2-form we shall present the explicit expressions for three vector valued 1-forms that define almost complex structure and further show that they are integrable as their Nijenhuis tensors vanish. In physics literature [1] the hyper-Kähler property of instantons was observed and used very effectively in some important examples of gravitational instanton metrics, whereas in the mathematical literature [2] hyper-Kähler structure is discussed in rather abstract terminology of a category of “words” which consist of combinations of three letters generating elements of the space of 2-forms. Here we shall use a systematic approach which is based on the Newman-Penrose formalism for Euclidean signature to provide explicit proof of hyper-Kähler structure when curvature is self-dual. Some of the results we shall present are not new, but they appear as part of a systematic exposition in the framework of the Newman-Penrose formalism which is a powerful technique that can be applied to other problems of current interest in physics and mathematics.

It is well-recognized that in general relativity the most useful formalism for investigating properties of physically interesting exact solutions of the Einstein field equations is the Newman-Penrose (NP) formalism [3] and this should be valid in the case of Euclidean signature as well. The original NP formalism takes advantage of the Lorentzian signature of spacetime in an essential way and is therefore not suitable for studying properties of gravitational instantons [4], [5] where the metric is strictly Riemannian. Goldblatt [6], [7] has developed Newman-Penrose and Geroch-Held-Penrose formalisms for Euclidean signature and we shall build on his work. The NP formalism becomes compact when everything is expressed in terms of differential forms appropriate to the NP basis, an approach developed by one of us in an unpublished work [8] some time ago. Now we shall present the differential form version of the NP formalism for Euclidean signature and in view of the fact that it is largely even unknown for the Lorentzian case, we shall develop the whole formalism *ab initio*. Finally, Goldblatt’s work is aimed towards the analysis of two-sided Ricci-flat metrics and he has specialized the identities of Riemannian geometry to this end. Our primary interest is in self-duality but we shall not specialize the identities of the NP formalism.

The essential outlines of a Newman-Penrose formalism and the underlying spinor structure of 4-dimensional Riemannian manifolds were considered by Penrose and Rindler [9] in their general consideration of complex manifolds. Plebanski [10] has developed it for complex space-time. These were followed up by the work of Flaherty [11], [12] who discussed in detail many of these issues, again, for 4-dimensional complex manifolds. Still another approach to complex spacetime is \mathcal{H} -space [13]. We shall be interested in the reality condition that selects real 4-dimensional manifolds with Euclidean signature and for this purpose it is more economical, in fact more insightful, to bypass the detour through complex spaces altogether and restrict the scope of our discussion from the outset to real manifolds. This is also the approach of Goldblatt [6]. Earlier Gibbons and Pope [14], [15] had used spinors in their discussion of gravitational instantons without developing the full NP formalism.

Recently it was recognized that a class of gravitational instantons can be put into one to one correspondence with minimal surfaces in 3-dimensional Euclidean space [16], [17]. In the case of Yang-Mills instantons such a correspondence was established by Comtet [18]. Among the gravitational instantons derived from minimal surfaces, the instanton that corresponds to the helicoid minimal surface has many interesting properties [19] and the Newman-Penrose formalism for positive definite signature is necessary in order to carry out a full investigation of the properties of this as well as all gravitational instantons.

The principal issue in the discussion of gravitational instantons is the decomposition of the gravitational field into its self-dual and anti-self-dual, alternatively left-half-flat and right-half-flat, parts and this decomposition arises naturally if one appeals to spinors. Since the Lorentz group is now replaced by $SO(4)$ which is isomorphic to $[SU(2) \times SU(2)]/Z_2$, *two independent* unitary linear transformations of the complex 2-dimensional plane correspond to proper transformations of 4-dimensional Euclidean space and vice versa. Therefore we need two independent sets of spin frames for a discussion of the spinor structure of 4-dimensional Riemannian manifolds with Euclidean signature. The necessity of dealing with two independent spin frames may seem cumbersome but it has the added bonus that in terms of these 2-component spinors the self-dual and anti-self dual properties of the gravitational field separate out very naturally. We shall show that the spin coefficients, connection 1-form and the curvature 2-form split up into two

independent sets corresponding to the two spin frames and each set can be readily identified as belonging to either one of the self-dual, or anti-self-dual sectors.

Complex structure plays a vitally important role in real 4-dimensional Riemannian manifolds that describe gravitational instantons but this role is often obscured by going through complex spacetime. We shall present the explicit expressions for three vector-valued 1-forms that define almost complex structures for 4-dimensional real Riemannian manifolds and furthermore show that they are all integrable when the curvature 2-form is self-dual. The condition for the integrability of these almost complex structures, namely the vanishing of their Nijenhuis tensors, is satisfied for a self-dual connection. In the case of gravitational instantons this condition can always be satisfied by an appropriate choice of gauge because curvature is self-dual. Hence we prove the theorem that gravitational instantons admit tri-complex, or hyper-Kähler structure. This property of gravitational instantons was earlier used to great effect in particular examples [1]. Goldblatt [6] has considered general expressions for almost complex structure and proposed two vector-valued 1-forms. However, they cannot both define almost complex structure simultaneously because one of them belongs to the self-dual, whereas the other to the anti-self-dual sector and therefore only one of them enters into hyper-Kähler structure. Salamon [2] has discussed hyper-Kähler structure, however, the proof of integrability of the three almost complex structures through a check of their Nijenhuis tensor is missing except in one trivial case which follows directly from the Kähler property itself. In sect.8 we shall present an explicit and systematic proof of hyper-Kähler structure of gravitational instantons using the Newman-Penrose formalism for Euclidean signature.

In order to present a complete account we shall derive the Ricci and Bianchi identities as well as the Maxwell equations in terms of spin coefficients, complex dyad scalars of curvature and electromagnetic field respectively, using the NP basis differential forms. All these identities of Riemannian geometry fall into two independent sets and each set can be obtained from the other by an operation, “tilde”, that swaps the spin frames. So, in effect, with the help of the tilde transformation we need to carry only one half of the identities of Riemannian geometry. We shall also discuss the spinor description of the topological invariants and Petrov classification of gravitational instantons where the three self-dual basis 2-forms which are the Kähler 2-forms for the three almost complex structures in hyper-Kähler

structure play the role of a triad basis. In the following we shall use the work of Goldblatt [6] repeatedly, however, we shall adopt it to differential forms which makes it compact and in the interest of clarity we have found it useful to develop the whole formalism *ab initio*.

2 Complex dyad

At each point on a 4-dimensional real Riemannian manifold we shall introduce an isotropic dyad, a pair of complex 4-vectors l^μ, m^μ which together with their complex conjugates serve to define a tetrad

$$e_a^\mu = \{l^\mu, \bar{l}^\mu, m^\mu, \bar{m}^\mu\} \quad (1)$$

where bar denotes complex conjugation. The inverse of the basis (1) is $e_\nu^a = \{\bar{l}_\nu, l_\nu, \bar{m}_\nu, m_\nu\}$ so that the co-frame 1-forms are given by

$$\{\bar{l}, l, \bar{m}, m\} = e_\nu^a dx^\nu = e^a \quad (2)$$

and the metric is expressed in the form

$$ds^2 = l \otimes \bar{l} + \bar{l} \otimes l + m \otimes \bar{m} + \bar{m} \otimes m \quad (3)$$

which is the analog of the double null form in the Newman-Penrose formalism. The metric has positive definite signature. The legs of the complex dyad satisfy the normalization conditions

$$\begin{aligned} l_\mu \bar{l}^\mu &= 1, & m_\mu \bar{m}^\mu &= 1, \\ l_\mu l^\mu &= 0, & m_\mu m^\mu &= 0, & l_\mu m^\mu &= 0, & l_\mu \bar{m}^\mu &= 0, \end{aligned} \quad (4)$$

where eqs.(4) express the fact that we have chosen an isotropic dyad. In the isotropic frame the metric will be given by

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (5)$$

which is manifest from (3). Further, we have the completeness relation

$$\delta_\nu^\mu = l^\mu \bar{l}_\nu + \bar{l}^\mu l_\nu + m^\mu \bar{m}_\nu + \bar{m}^\mu m_\nu \quad (6)$$

which follows from the choice of the complex dyad (1) and its inverse. Next we define the intrinsic derivative operators, or the directional derivative along the legs of the complex dyad. Following the Newman-Penrose notation we have

$$D = l^\mu \frac{\partial}{\partial x^\mu}, \quad \delta = m^\mu \frac{\partial}{\partial x^\mu} \quad (7)$$

together with their complex conjugates and inserting the completeness relation (6) into the definition of the exterior derivative we find

$$d = \bar{l} D + l \bar{D} + m \bar{\delta} + \bar{m} \delta \quad (8)$$

as the resolution of the exterior derivative along the legs of this complex dyad.

3 Spin frames

In the definition of the legs of the tetrad there is a degree of freedom, a global gauge freedom, of rotation. This is given by $SO(4)$ which can be decomposed into a product of two independent $SU(2)$ degrees of freedom. The action of each $SU(2)$ can be described by a linear fractional transformation of the compactified complex plane, that is, the Riemann sphere and the spin frame is the familiar spinor basis. Thus we begin by introducing two spin frames with bases

$$\zeta_a^A = \{ o^A, \iota^A \} \quad A = 1, 2 \quad a = 0, 1 \quad (9)$$

and

$$\tilde{\zeta}_{x'}^{X'} = \{ \tilde{o}^{X'}, \tilde{\iota}^{X'} \} \quad X' = 1', 2' \quad x' = 0', 1' \quad (10)$$

of two independent 2-component spinors respectively. Here capital Latin indices refer to the components of spinors and small Latin indices run over the two spinors, omicron and iota, respectively. Both sets of these indices will be raised and lowered by the Levi-Civita symbol from the right. Thus we have $o_A = o^B \epsilon_{BA} = -\epsilon_{AB} o^B$ and the normalization conditions are

$$o_A \iota^A = 1, \quad \tilde{o}_{X'} \tilde{\iota}^{X'} = 1 \quad (11)$$

with all other contractions vanishing identically. It is not possible to contract the unprimed and primed indices as they refer to objects that belong to

different spaces. It may seem redundant to use tilde over spinors when this information is already carried by whether or not they carry primed indices, however, as we shall soon see tilde is a very important operation and it will be useful to keep it. Suitable combinations of spinors from the bases (9) and (10) will determine the complex dyad through the Infeld-van der Waerden [20] connecting quantities $\sigma_{AX'}^\mu$. Without loss of generality we may choose the representation

$$\begin{aligned} l^\mu &= \sigma_{00'}^\mu = \sigma_{AX'}^\mu o^A \tilde{o}^{X'} \\ \bar{l}^\mu &= \sigma_{11'}^\mu = \sigma_{AX'}^\mu t^A \tilde{t}^{X'} \\ m^\mu &= \sigma_{01'}^\mu = \sigma_{AX'}^\mu o^A \tilde{t}^{X'} \\ \bar{m}^\mu &= -\sigma_{10'}^\mu = -\sigma_{AX'}^\mu t^A \tilde{o}^{X'} \end{aligned} \quad (12)$$

which satisfies the normalization conditions (4) and (11). We may summarize these definitions in the form of a matrix

$$\sigma_{AX'}^\mu = \begin{pmatrix} l^\mu & m^\mu \\ -\bar{m}^\mu & \bar{l}^\mu \end{pmatrix}, \quad \sigma_\mu^{AX'} = \begin{pmatrix} \bar{l}_\mu & \bar{m}_\mu \\ -m_\mu & l_\mu \end{pmatrix} \quad (13)$$

which determine the correspondence between tensor and spinor fields. For example, the components of the metric are given by

$$g_{\mu\nu} = \epsilon_{AB} \epsilon_{X'Y'} \sigma_\mu^{AX'} \sigma_\nu^{BY'} \quad (14)$$

and we shall write it simply as

$$\begin{aligned} g_{\mu\nu} &\leftrightarrow g_{AX'BY'} = \epsilon_{AB} \epsilon_{X'Y'} \\ &\Leftrightarrow \epsilon_{AB} \end{aligned} \quad (15)$$

following conventional usage. The Infeld-van der Waerden connecting quantities satisfy

$$\delta_\mu^\nu = \sigma_\mu^{AX'} \sigma_{AX'}^\nu, \quad \sigma_{AX'}^\mu \sigma_\mu^{BY'} = \delta_A^B \delta_{X'}^{Y'} \quad (16)$$

which reproduce the completeness relation (6). In dealing with 2-component spinors we shall repeatedly use the basic spinor identity in two dimensions

$$\epsilon_A[B \epsilon_{C D}] = 0 \quad (17)$$

where square parantheses denote skew symmetrization. As a consequence we have the familiar relation

$$\xi_{[AB]} = \frac{1}{2} \xi_C{}^C \epsilon_{AB} \quad (18)$$

where ξ_{AB} is an arbitrary second rank 2-component spinor.

4 Ricci rotation coefficients

The Ricci rotation coefficients are the complex dyad components of the Levi-Civita connection which are defined by

$$\gamma_{ijk} = e_{j\mu;\nu} e_i^\mu e_k^\nu = -\gamma_{jik} \quad (19)$$

where semicolon denotes covariant differentiation. For the basis (1) the Ricci rotation coefficients will be labelled as

$$\begin{aligned} \kappa &= \gamma_{311} = l_{\mu;\nu} m^\mu l^\nu, & \pi &= \gamma_{231} = m_{\mu;\nu} \bar{l}^\mu l^\nu, \\ \tau &= \gamma_{312} = l_{\mu;\nu} m^\mu \bar{l}^\nu, & \nu &= \gamma_{232} = m_{\mu;\nu} \bar{l}^\mu \bar{l}^\nu, \\ \sigma &= \gamma_{313} = l_{\mu;\nu} m^\mu m^\nu, & \mu &= \gamma_{233} = m_{\mu;\nu} \bar{l}^\mu m^\nu, \\ \rho &= -\gamma_{314} = l_{\mu;\nu} m^\mu \bar{m}^\nu, & \lambda &= -\gamma_{234} = m_{\mu;\nu} \bar{l}^\mu \bar{m}^\nu, \end{aligned} \quad (20)$$

$$\begin{aligned} \epsilon &= \frac{1}{2}(\gamma_{211} - \gamma_{341}) = \frac{1}{2}(l_{\mu;\nu} \bar{l}^\mu l^\nu - \bar{m}_{\mu;\nu} m^\mu l^\nu), \\ \gamma &= \frac{1}{2}(\gamma_{212} + \gamma_{342}) = \frac{1}{2}(l_{\mu;\nu} \bar{l}^\mu \bar{l}^\nu + \bar{m}_{\mu;\nu} m^\mu \bar{l}^\nu), \\ \alpha &= -\frac{1}{2}(\gamma_{214} - \gamma_{344}) = -\frac{1}{2}(l_{\mu;\nu} \bar{l}^\mu \bar{m}^\nu - \bar{m}_{\mu;\nu} m^\mu \bar{m}^\nu), \\ \beta &= \frac{1}{2}(\gamma_{213} + \gamma_{343}) = \frac{1}{2}(l_{\mu;\nu} \bar{l}^\mu m^\nu + \bar{m}_{\mu;\nu} m^\mu m^\nu), \end{aligned}$$

again following a notation as close as possible to the Newman-Penrose spin coefficients for Lorentzian signature.

The most convenient way of calculating the Ricci rotation coefficients is through the use of differential forms [8]. Taking the exterior derivative of the basis 1-forms (2) and expressing the result in terms of the basis 2-forms we obtain

$$\begin{aligned} dl &= (\bar{\gamma} - \epsilon) l \wedge \bar{l} + (\alpha + \bar{\beta} - \bar{\pi}) l \wedge m + (\tau - \beta - \bar{\alpha}) l \wedge \bar{m} \\ &\quad - \bar{\nu} \bar{l} \wedge m + \kappa \bar{l} \wedge \bar{m} - (\bar{\lambda} + \rho) m \wedge \bar{m} \\ dm &= (\pi + \tau) \bar{l} \wedge l - (\bar{\epsilon} + \gamma - \lambda) l \wedge m - \mu l \wedge \bar{m} \\ &\quad + (\epsilon - \rho + \bar{\gamma}) \bar{l} \wedge m + \sigma \bar{l} \wedge \bar{m} - (\bar{\alpha} - \beta) m \wedge \bar{m} \end{aligned} \quad (21)$$

where the coefficients are linear algebraic equations for the Ricci rotation coefficients. Thus the exterior derivative of the basis 1-forms l, m yields the Ricci rotation coefficients by a simple comparison of the result with eqs.(21).

5 Spin coefficients

With the choice of two independent spin-frames we necessarily arrive at two sets of spin coefficients which are defined by

$$\Gamma_{abAX'} = \zeta_{aB;AX'} \zeta_b^B, \quad \tilde{\Gamma}_{x'y'AX'} = \tilde{\zeta}_{x'Y';AX'} \tilde{\zeta}_{y'}^{Y'} \quad (22)$$

where $;\mu \leftrightarrow ;AX'$ denotes the spinor equivalent of the covariant derivative. This is a complex holomorphic operator since the primed and unprimed indices are independent. We note the symmetry in the first pair of complex dyad indices in both $\Gamma_{abAX'}$ and $\tilde{\Gamma}_{x'y'AX'}$ so that their traces will vanish. The spin coefficients are given in terms of the Ricci rotation coefficients. It is clear that due to the symmetries of $\Gamma_{abAX'}$ and its tilde counterpart, one needs to evaluate only six complex spin coefficients in each of the two cases. We find that $\Gamma_{a\ cd'}^b$ and $\tilde{\Gamma}_{x'\ cd'}^{y'}$, where indices are raised by the 2-dimensional Levi-Civita symbol, can be listed according to the table:

$\begin{matrix} b \\ a \\ cd' \end{matrix}$	$\begin{matrix} 0 \\ 0 \end{matrix}$	$\begin{matrix} 1 \\ 0 \end{matrix}$	$\begin{matrix} 0 \\ 1 \end{matrix}$	$\begin{matrix} y' \\ x' \end{matrix}$	$\begin{matrix} 0' \\ 0' \end{matrix}$	$\begin{matrix} 1' \\ 0' \end{matrix}$	$\begin{matrix} 0' \\ 1' \end{matrix}$
00'	ϵ	$-\kappa$	$\bar{\tau}$		$-\bar{\gamma}$	$-\bar{\nu}$	π
10'	α	$-\rho$	$-\bar{\sigma}$		$\bar{\beta}$	$\bar{\mu}$	λ
01'	$\bar{\alpha}$	$-\sigma$	$-\bar{\rho}$		β	$\bar{\lambda}$	μ
11'	$-\bar{\epsilon}$	$-\tau$	$\bar{\kappa}$		γ	$-\bar{\pi}$	ν

Table of Spin Coefficients

As we have remarked earlier, the most convenient way obtaining the spin coefficients is through eqs.(21).

5.1 Swapping symmetry

A glance at the table of spin coefficients suggests the introduction of “swapping symmetry,” namely tilde, which is an operation that turns all quantities with tilde into those without and vice versa. In other words tilde swaps the unprimed and primed spin frames, alternatively the two $SU(2)$ components of $SO(4)$. The significance of the tilde operation will emerge as the operation that turns self-dual into anti-self-dual parts of the curvature 2-form. Explicitly the tilde operation is given by the replacement of either one of the complex dyad vectors $l \leftrightarrow \bar{l}$, or $m \leftrightarrow \bar{m}$. For the sake of definiteness we shall use the tilde operation given by

$$l \leftrightarrow \bar{l} \quad (23)$$

with m and \bar{m} unchanged. In terms of the basic spinors this amounts to the replacement

$$o^A \mapsto i \tilde{l}^{X'}, \quad \iota^A \mapsto i \tilde{o}^{X'}, \quad \tilde{o}^{X'} \mapsto -i \iota^A, \quad \tilde{l}^{X'} \mapsto -i o^A \quad (24)$$

which preserves the normalization conditions (11). From eq.(12) it is seen that that the tilde operation given in eqs.(23) - (24) in turn implies the correspondence

$$0 \leftrightarrow 1', \quad 1 \leftrightarrow 0' \quad (25)$$

between the spinor indices. The action of the tilde operation on spin coefficients yields

$$\begin{array}{lll} \tau \leftrightarrow -\pi, & \epsilon \leftrightarrow -\gamma, & \alpha \leftrightarrow -\bar{\beta}, \\ \kappa \leftrightarrow -\nu, & \rho \leftrightarrow -\lambda, & \sigma \leftrightarrow -\mu \end{array} \quad (26)$$

which is immediate from the definitions (20). In fact, we could have halved the names of spin coefficients by using the tilde symbol but it is messy to deal with the complex conjugate of tilde. Finally, we note that under the simultaneous interchange of the legs of the complex dyad $l \leftrightarrow \bar{l}$, $m \leftrightarrow \bar{m}$ the result is simply the operation of complex conjugation which leaves the metric invariant.

5.2 Commutator identities

The commutator of different intrinsic derivative operators is given by the Lie bracket

$$[\partial_j, \partial_k] = C^i{}_{jk} \partial_i \quad (27)$$

where the structure functions are given by

$$C^i{}_{jk} = \gamma^i{}_{kj} - \gamma^i{}_{jk}$$

in terms of the Ricci rotation coefficients. Explicitly we have the following set of commutator identities

$$D\bar{D} - \bar{D}D = (\bar{\epsilon} - \gamma)D + (\bar{\gamma} - \epsilon)\bar{D} + (\bar{\pi} + \bar{\tau})\delta - (\pi + \tau)\bar{\delta} \quad (28)$$

$$\delta\bar{\delta} - \bar{\delta}\delta = (\lambda + \bar{\rho})D - (\bar{\lambda} - \rho)\bar{D} + (\alpha - \bar{\beta})\delta - (\bar{\alpha} - \beta)\bar{\delta} \quad (29)$$

$$D\delta - \delta D = (\pi - \beta - \bar{\alpha})D - \kappa\bar{D} + (\epsilon + \bar{\gamma} - \bar{\lambda})\delta - \sigma\bar{\delta} \quad (30)$$

$$\bar{D}\delta - \delta\bar{D} = (\bar{\alpha} + \beta - \tau)\bar{D} - (\bar{\epsilon} + \gamma - \bar{\rho})\delta + \nu D + \mu\bar{\delta} \quad (31)$$

which along with their complex conjugates consist of all six identities as the first two are pure imaginary. We note that under the tilde operation the pure imaginary identities (28) and (29) remain invariant, whereas (30) and (31) tranform into each other.

6 Connection 1-forms

The connection is an \mathcal{SO}_4 -valued 1-form which satisfies Cartan's equations of structure

$$de^\alpha + \omega^\alpha{}_\beta \wedge e^\beta = 0, \quad (32)$$

where the matrix of connection 1-forms is skew

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} \quad (33)$$

and due to this anti-symmetry, using the basic identity (18), the spinor equivalent of the full connection 1-form

$$\begin{aligned} \omega_{\alpha\beta} &\leftrightarrow \Gamma_{ax' by'} \\ &= \Gamma_{ab} \epsilon_{x'y'} + \tilde{\Gamma}_{x'y'} \epsilon_{ab} \\ &\Leftrightarrow \Gamma_{ab} \end{aligned} \quad (34)$$

naturally splits up into two pieces appropriate to the two sets of spin frames. Here Γ_{ab} and $\tilde{\Gamma}_{x'y'}$ are symmetric matrices of 1-forms and raising an index with the appropriate 2-dimensional Levi-Civita symbol we have

$$\Gamma_a{}^b = \Gamma_a{}^b{}_{cx'} \sigma_\mu^{cx'} dx^\mu, \quad \tilde{\Gamma}_{x'}{}^{y'} = \tilde{\Gamma}_{x'}{}^{y'}{}_{az'} \sigma_\mu^{az'} dx^\mu \quad (35)$$

which are traceless. From eqs.(22), (13), (20) and the table of spin coefficients we find that

$$\begin{aligned}\Gamma_0^0 &= \frac{1}{2} \left(l_{\mu;\nu} \bar{l}^\mu + m_{\mu;\nu} \bar{m}^\mu \right) dx^\nu = \epsilon \bar{l} - \bar{\epsilon} l - \alpha m + \bar{\alpha} \bar{m} \\ \Gamma_0^1 &= -l_{\mu;\nu} m^\mu dx^\nu = -\tau l - \kappa \bar{l} + \rho m - \sigma \bar{m} \\ \Gamma_1^0 &= -\bar{\Gamma}_0^1, \quad \Gamma_1^1 = -\Gamma_0^0\end{aligned}\tag{36}$$

and

$$\begin{aligned}\tilde{\Gamma}_{0'}^{0'} &= \frac{1}{2} \left(l_{\mu;\nu} \bar{l}^\mu - m_{\mu;\nu} \bar{m}^\mu \right) dx^\nu = \gamma l - \bar{\gamma} \bar{l} + \beta \bar{m} - \bar{\beta} m \\ \tilde{\Gamma}_{0'}^{1'} &= l_{\mu;\nu} \bar{m}^\mu dx^\nu = -\bar{\pi} l - \bar{\nu} \bar{l} - \bar{\mu} m + \bar{\lambda} \bar{m} \\ \tilde{\Gamma}_{1'}^{0'} &= -\bar{\tilde{\Gamma}}_{0'}^{1'}, \quad \tilde{\Gamma}_{1'}^{1'} = -\tilde{\Gamma}_{0'}^{0'}\end{aligned}\tag{37}$$

provide the explicit expression for the connection 1-forms. There are important differences here with the Newman-Penrose formalism for Lorentzian signature. First of all we have two sets of connection 1-forms in place of a single set. These two sets of connection 1-forms necessarily exhibit a “redundancy” in that their off-diagonal elements are related. This is expected since we are dealing with two spin frames and setting up two sets of connection 1-forms, whereas the number of independent spin coefficients is fixed by the number of dimensions of the manifold. This redundancy is actually a blessing in disguise because eventually it will prove to be an enormous convenience in our considerations of self-dual and anti-self-dual 2-forms. Finally, we observe that

$$d\sigma^{ax'} = -\Gamma_b^a \wedge \sigma^{bx'} - \tilde{\Gamma}_{y'}^{x'} \wedge \sigma^{ay'}\tag{38}$$

is the compact form of eqs.(21). Under the tilde operation Γ simply goes over into $\tilde{\Gamma}$ according to the rules (23) - (26).

6.1 Self-dual gauge

A most convenient but not necessary device in dealing with 4-dimensional Riemannian manifolds that admit (anti)-self-dual curvature is to pick a frame where the connection itself shares this property. We shall now show that the two independent sets of connection 1-forms in eqs.(36) and (37) determine the self-dual and anti-self-dual connection 1-forms. With the definition

$$\pm \omega_{\alpha\beta} = \frac{1}{2} \left(\omega_{\alpha\beta} \pm \frac{1}{2} \epsilon_{\alpha\beta}^{\gamma\delta} \omega_{\gamma\delta} \right),\tag{39}$$

the self-dual and anti-self dual sets of connection 1-forms are given by

$${}^-\omega_{\alpha\beta} = 0, \quad {}^+\omega_{\alpha\beta} = 0, \quad (40)$$

respectively. For the spinor equivalent of ${}^\pm\omega_{\alpha\beta}$ we need the spinor equivalent of the totally anti-symmetric Levi-Civita alternating symbol

$$\epsilon_{\alpha\beta\gamma\delta} \leftrightarrow \epsilon_{ax'by'cz'dw'} = \epsilon_{ac} \epsilon_{bd} \epsilon_{x'w'} \epsilon_{y'z'} - \epsilon_{ad} \epsilon_{bc} \epsilon_{a'z'} \epsilon_{b'd'} \quad (41)$$

which enters into eq.(39) and we find

$$\begin{aligned} {}^-\omega_{ax'by'} &= {}^-\omega_{(ab)[x'y']} = \Gamma_{ab}\epsilon_{x'y'}, \\ {}^+\omega_{ax'by'} &= {}^+\omega_{[ab](x'y')} = \Gamma_{x'y'}\epsilon_{ab} \end{aligned} \quad (42)$$

respectively. We conclude that the necessary and sufficient conditions for the connection 1-form to be self-dual, or anti-self-dual are given by

$$\Gamma_{ab} \equiv 0, \quad \tilde{\Gamma}_{x'y'} \equiv 0 \quad (43)$$

respectively. Returning back to eqs.(36) and (37) we note that the condition for self-duality of the connection implies

$$\epsilon = \alpha = \tau = \kappa = \rho = \sigma = 0, \quad (44)$$

while the criterion for anti-self-duality requires that

$$\gamma = \beta = \pi = \mu = \nu = \lambda = 0. \quad (45)$$

Given a metric with (anti)-self-dual curvature 2-form it is always possible to choose a gauge, namely frame for which either eqs.(44), or (45) hold.

7 Basis 2-forms

The spinor equivalent of the basis 2-forms are wedge products of the Infeld-van der Waerden matrices of basis 1-forms (13)

$$e_\mu^a e_\nu^b dx^\mu \wedge dx^\nu \leftrightarrow \sigma^{ax'} \wedge \sigma^{by'}$$

and we have two sets of basis 2-forms

$$L_a^b = \frac{1}{2} \sigma_{\mu ax'} \sigma_{\nu}^{bx'} dx^\mu \wedge dx^\nu = \frac{1}{2} \sigma_{ax'} \wedge \sigma^{bx'} \quad (46)$$

$$\tilde{L}_{x'}^{y'} = \frac{1}{2} \sigma_{ax'} \wedge \sigma^{ay'} \quad (47)$$

corresponding to the two spin frames. These are consequences of the spinor relation

$$L_{ax'}^{by'} = \sigma_{ax'} \wedge \sigma^{by'} = L_a^b \epsilon_{x'}^{y'} + \tilde{L}_{x'}^{y'} \epsilon_a^b \quad (48)$$

due to skew symmetry in ax' , by' and the basic spinor identity (18). Taking the components of eqs.(46) and (47) we find

$$\begin{aligned} L_0^0 &= \frac{1}{2} (l \wedge \bar{l} + m \wedge \bar{m}), & \tilde{L}_{0'}^{0'} &= \frac{1}{2} (l \wedge \bar{l} - m \wedge \bar{m}), \\ L_0^1 &= -l \wedge m, & \tilde{L}_{0'}^{1'} &= l \wedge \bar{m}, \\ L_1^0 &= \bar{l} \wedge \bar{m}, & \tilde{L}_{1'}^{0'} &= -\bar{l} \wedge m \end{aligned} \quad (49)$$

and the remaining components follow from the vanishing of the trace. We note that once again $L_0^1 = -\bar{L}_1^0$ and $\tilde{L}_{0'}^{1'} = -\tilde{\bar{L}}_{1'}^{0'}$ as in the case of eqs.(36), (37) and this property will hold for the curvature 2-forms as well.

The decomposition of the basis 2-forms into self-dual and anti-self-dual parts is immediate. Using the definition of Hodge star and the completeness relation (6), we find that

$$*L_a^b = -L_a^b \quad * \tilde{L}_{x'}^{y'} = \tilde{L}_{x'}^{y'} \quad (50)$$

so that the two sets of independent basis 2-forms (46), (47) can be recognized as the self-dual and anti-self-dual objects. Finally, we note that these 2-forms satisfy

$$\begin{aligned} d L_a^b + L_a^c \wedge \Gamma_c^b - \Gamma_a^c \wedge L_c^b &= 0 \\ d \tilde{L}_{x'}^{y'} + \tilde{L}_{x'}^{z'} \wedge \tilde{\Gamma}_{z'}^{y'} - \tilde{\Gamma}_{x'}^{z'} \wedge \tilde{L}_{z'}^{y'} &= 0 \end{aligned} \quad (51)$$

which may be called the “zeroth” Bianchi identities but are better known as Ricci’s lemma. Their verification is immediate from eqs.(49) and (36)-(37). Hence L_a^b and $\tilde{L}_{x'}^{y'}$ form the spinor equivalent of the self-dual and anti-self-dual basis of the space of 2-forms respectively and their covariant derivatives vanish identically. In eqs.(51) we see a phenomenon which will

appear repeatedly, namely, we need to write down only one half of the “zeroth” Bianchi identities and remember that the same equation is valid for its tilde version with primed indices as well. In other words these equations hold for both sets of spin frames independently.

8 Complex structure

The complex dyad formalism and the corresponding spinor structure for 4-dimensional real strictly Riemannian manifolds provides a very natural framework for the discussion of the complex structure of gravitational instantons in its full generality. A real even-dimensional manifold will admit almost complex structure provided there exists a real differentiable vector-valued 1-form

$$J = J_\mu{}^\nu dx^\mu \otimes \frac{\partial}{\partial x^\nu} \quad (52)$$

which is an anti-involution with the components of J satisfying

$$J_\mu{}^\rho J_\rho{}^\nu = -\delta_\mu^\nu \quad (53)$$

so that

$$J[J(\xi)] = -\xi \quad (54)$$

for any differentiable vector field ξ . The structure functions of the almost complex structure must be real, that is

$$J_\mu{}^\alpha J_\nu{}^\beta g_{\alpha\beta} = g_{\mu\nu} \quad (55)$$

which is the Hermitian property. From eqs.(53) and (55) it follows that lowering the vector index with the metric we get a skew-symmetric tensor

$$J_{\mu\nu} = -J_{\nu\mu} \quad (56)$$

and in this connection it is worth recalling that almost complex structure is a metric-independent concept but of course in our case there exists a Riemannian metric and we shall use it.

The explicit expression for almost complex structure of a general gravitational instanton metric that admits self-dual curvature 2-form assumes a

simple form in the complex dyad (1). There exists three such vector valued 1-forms, which is implicit in Salamon [2], that together will form hyper-Kähler structure, namely

$$\begin{aligned}
J_{1\mu}{}^\nu &= -i \left(l_\mu m^\nu - m_\mu l^\nu - \bar{l}_\mu \bar{m}^\nu + \bar{m}_\mu \bar{l}^\nu \right) \\
J_{2\mu}{}^\nu &= l_\mu m^\nu - m_\mu l^\nu + \bar{l}_\mu \bar{m}^\nu - \bar{m}_\mu \bar{l}^\nu \\
J_{3\mu}{}^\nu &= -i \left(l_\mu \bar{l}^\nu - \bar{l}_\mu l^\nu + m_\mu \bar{m}^\nu - \bar{m}_\mu m^\nu \right)
\end{aligned} \tag{57}$$

which can readily be verified to be real and satisfy all the properties required of an almost complex structure, as expressed by eqs.(53), (55) and (56). Furthermore, they transform properly under $SU(2)$ whereas in general only tensorial transformation properties under $SO(4)$ would be required. Hyper-Kähler structure where

$$J_{i\mu}{}^\sigma J_{j\sigma}{}^\nu = -J_{j\mu}{}^\sigma J_{i\sigma}{}^\nu = \epsilon_{ijk} J_{k\mu}{}^\nu \tag{58}$$

is responsible for this. Goldblatt [6] had proposed the almost complex structure $J_{3\mu}{}^\nu$ above and another one that differs from it by the choice of opposite sign in the last two terms. However, these two choices belong to different sectors of self-duality and they cannot coexist in hyper-Kähler structure.

The structure functions of these three almost complex structures can be used to define 2-forms $\omega_i = \frac{1}{2} J_{i\mu\nu} dx^\mu \wedge dx^\nu$, namely

$$\begin{aligned}
\omega_1 &= -i (l \wedge m - \bar{l} \wedge \bar{m}) \\
\omega_2 &= l \wedge m + \bar{l} \wedge \bar{m} \\
\omega_3 &= -i (l \wedge \bar{l} + m \wedge \bar{m})
\end{aligned} \tag{59}$$

which are simply the three real self-dual basis 2-forms of eqs.(49). In particular, ω_3 is the Kähler 2-form. We have already remarked that the connection 1-form for gravitational instantons can be chosen such that the gauge is self-dual which is given by eqs.(44). Thus the 2-forms (59) are closed by virtue of “zeroth” Bianchi identities (51), or this can be demonstrated explicitly using eqs.(21). So from the structure functions of the three almost complex structures we obtain the hyper-Kähler 2-forms. The spinor equivalent of almost complex structure assumes the form

$$J_{i\mu\nu} \leftrightarrow J_{iAB} \epsilon_{X'Y'} + \tilde{J}_{iX'Y'} \epsilon_{AB} \tag{60}$$

where the essential element, namely the second rank symmetric spinors J_{iAB} and $\tilde{J}_{iX'Y'}$ are simple bi-spinors

$$\begin{aligned} J_{1AB} &= -i(o_A o_B - \iota_A \iota_B) \\ J_{2AB} &= o_A o_B + \iota_A \iota_B \\ J_{3AB} &= 2i o_{(A} \iota_{B)} \end{aligned} \tag{61}$$

constructed from the basis spinors. Similar relations hold for primed indices.

The question now arises as to whether or not these almost complex structure are integrable which is the necessary and sufficient condition for hyper-Kähler structure. The integrability condition is given by the vanishing of the Nijenhuis tensor [22]

$$N_{\mu\nu}{}^\alpha = J_\mu{}^\sigma (J_{\nu;\sigma}{}^\alpha - J_{\sigma;\nu}{}^\alpha) - J_\nu{}^\sigma (J_{\mu;\sigma}{}^\alpha - J_{\sigma;\mu}{}^\alpha) \tag{62}$$

which is a vector-valued 2-form. It will be convenient to define both a vector-valued 1-form \mathcal{J}^μ and a form-valued vector field \mathbf{J}_ν

$$\mathcal{J}^\mu \equiv J^\mu_\nu dx^\nu, \quad \mathbf{J}_\nu \equiv J^\mu_\nu \frac{\partial}{\partial x^\mu} \tag{63}$$

in terms of which the Nijenhuis vector-valued 2-form can be written as

$$N^\alpha = \frac{1}{2} N^\alpha_{\mu\nu} dx^\mu \wedge dx^\nu = dx^\nu \wedge i_{\mathbf{J}_\nu} \{d\mathcal{J}^\alpha\} \tag{64}$$

where i denotes contraction of the 2-form $d\mathcal{J}^\alpha$ with the vector field \mathbf{J}_ν . The Nijenhuis tensor is metric-independent because the connection washes out in eq.(62) so that all the covariant derivatives can be replaced by partial derivatives.

In our case the three Nijenhuis vector-valued 2-forms are given by

$$\begin{aligned} N_1^\alpha &= [(\bar{\rho} + 2\bar{\epsilon} - \sigma) l^\alpha + (\kappa + 2\alpha + \bar{\tau}) m^\alpha] (l \wedge \bar{l} + m \wedge \bar{m}) \\ &\quad + [(\tau + \bar{\kappa} + 2\bar{\alpha}) l^\alpha + (\bar{\sigma} - \rho - 2\epsilon) m^\alpha] (l \wedge m + \bar{l} \wedge \bar{m}) + cc \\ N_2^\alpha &= [(\bar{\rho} + 2\bar{\epsilon} + \sigma) l^\alpha + (2\alpha + \bar{\tau} - \kappa) m^\alpha] (l \wedge \bar{l} + m \wedge \bar{m}) \\ &\quad + [(\bar{\kappa} - \tau - 2\bar{\alpha}) l^\alpha + (\rho + \bar{\sigma} + 2\epsilon) m^\alpha] (l \wedge m - \bar{l} \wedge \bar{m}) + cc \\ N_3^\alpha &= 4(\bar{\kappa} l^\alpha + \bar{\sigma} m^\alpha) l \wedge m + cc \end{aligned} \tag{65}$$

where cc denotes the complex conjugate of the foregoing terms and the condition for all of these almost complex structures to be integrable requires that the coefficients of vector valued 2-forms in eqs.(65) must all vanish. As it was noted by Salamon [2], $N_3^\alpha = 0$ is automatically satisfied for Kähler metrics, however, he has not directly checked the vanishing of the Nijenhuis tensor for the remaining two almost complex structures. From eqs.(65) we arrive the conditions

$$\epsilon = \alpha = \tau = \kappa = \rho = \sigma = 0, \quad (66)$$

for hyper-Kähler structure. But this is precisely the same as the condition for a self-dual connection given by eqs.(44). Since gravitational instantons admit self-dual curvature we can always find a frame where the connection is self-dual. Thus we have

Theorem

Gravitational instantons admit hyper-Kähler, or tri-complex structure. The almost complex structures given by eqs.(57) are all integrable for gravitational instantons.

9 Curvature

Complex dyad scalars formed from the components of the Riemann curvature tensor in the frame (1) are physically important quantities. As in the Newman-Penrose formalism with Lorentzian signature it will be useful to start with a list of curvature scalars adopted to this tetrad. First we have the usual decomposition of the Riemannian tensor

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= C_{\mu\nu\rho\sigma} + \frac{1}{2} (g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma}) \\ &\quad - \frac{1}{6} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R \end{aligned} \quad (67)$$

where $C_{\mu\nu\rho\sigma}$ is the conformal Weyl tensor, $R_{\mu\nu} = g^{\rho\sigma}R_{\mu\rho\nu\sigma}$ is the Ricci tensor and R is the scalar of curvature. For the complex dyad (1) we introduce two sets of Weyl scalars

$$\begin{aligned} \Psi_0 &= C_{1313} = C_{\mu\nu\sigma\tau}l^\mu m^\nu l^\sigma m^\tau & \tilde{\Psi}_0 &= C_{1414} = C_{\mu\nu\sigma\tau}l^\mu \bar{m}^\nu l^\sigma \bar{m}^\tau \\ \Psi_1 &= C_{1213} = C_{\mu\nu\sigma\tau}l^\mu \bar{l}^\nu l^\sigma m^\tau & \tilde{\Psi}_1 &= C_{1241} = C_{\mu\nu\sigma\tau}l^\mu \bar{l}^\nu \bar{m}^\tau l^\sigma \\ \Psi_2 &= C_{1324} = C_{\mu\nu\sigma\tau}l^\mu m^\nu \bar{l}^\sigma \bar{m}^\tau & \tilde{\Psi}_2 &= C_{1423} = C_{\mu\nu\sigma\tau}l^\mu \bar{m}^\nu \bar{l}^\sigma m^\tau \end{aligned} \quad (68)$$

which consist of a total of ten complex dyad scalars as Weyl scalars with subscript 2 are real. In addition we also introduce the trace-free Ricci scalars

$$\begin{aligned}\Phi_{00} &= -\frac{1}{2} R_{\mu\nu} l^\mu l^\nu = \bar{\Phi}_{22}, & \Phi_{01} &= -\frac{1}{2} R_{\mu\nu} l^\mu m^\nu = -\bar{\Phi}_{21}, \\ \Phi_{02} &= -\frac{1}{2} R_{\mu\nu} m^\mu m^\nu = \bar{\Phi}_{20}, & \Phi_{10} &= \frac{1}{2} R_{\mu\nu} l^\mu \bar{m}^\nu = -\bar{\Phi}_{12}, \\ \Phi_{11} &= -\frac{1}{2} R_{\mu\nu} l^\mu \bar{l}^\nu + 3\Lambda\end{aligned}\quad (69)$$

where Φ_{11} is real and thus we have the nine independent trace-free Ricci scalars. Together with the scalar of curvature

$$\Lambda = \frac{1}{24} R \quad (70)$$

the complex dyad scalars (68) and (69) determine the required twenty independent components of the Riemann tensor.

Following the general scheme of Penrose we shall obtain the curvature spinors which are the spinor equivalent of the Riemann curvature tensor (67). The decomposition of the Riemann curvature spinor into Weyl, trace-free Ricci and the scalar of curvature spinors in terms of 2-component spinors brings out the meaning of the curvature complex dyad scalars in the clearest way. Using the symmetry properties of the Riemann tensor, repeated applications of the basic spinor identity (18) yield

$$\begin{aligned}R_{\alpha\beta\gamma\delta} &\leftrightarrow R_{AX'BY'CZ'DW'} \\ &= X_{ABCD} \epsilon_{X'Y'} \epsilon_{Z'W'} + \tilde{X}_{X'Y'Z'W'} \epsilon_{AB} \epsilon_{CD} \\ &\quad + \Phi_{ABZ'W'} \epsilon_{X'Y'} \epsilon_{CD} + \tilde{\Phi}_{X'Y'CD} \epsilon_{AB} \epsilon_{Z'W'}\end{aligned}\quad (71)$$

where we have introduced the curvature spinors

$$X_{ABCD} = \frac{1}{4} R_{ABM'}{}^{M'}{}_{CDN'}{}^{N'}, \quad \Phi_{ABX'Y'} = \frac{1}{4} R_{ABM'}{}^{M'}{}_{N'}{}^N{}_{X'Y'} \quad (72)$$

together with their tilde counterparts. The anti-symmetry of the curvature tensor in its first and last pair of indices gives rise to the relations

$$X_{ABCD} = X_{(AB)(CD)}, \quad \Phi_{ABX'Y'} = \Phi_{(AB)(X'Y')} \quad (73)$$

and similarly for the tildes. The symmetry of the Riemann tensor under the interchange of the first and last pair of indices is reflected by

$$X_{ABCD} = X_{CDAB}, \quad \Phi_{ABX'Y'} = \tilde{\Phi}_{ABX'Y'} \quad (74)$$

in the the curvature spinors. It follows that only the curvature spinors X_{ABCD} , $\tilde{X}_{X'Y'Z'W'}$ and $\Phi_{ABX'Y'}$ are independent. As a consequences of these relations we have

$$X_{ABC}{}^B = 3\Lambda \epsilon_{AC} \quad (75)$$

and $\Lambda = \tilde{\Lambda}$, cf eq.(70). Using these results and also the basic spinor identity (18) we can write the totally symmetric part of X_{ABCD} as follows

$$X_{ABCD} = \Psi_{ABCD} + \Lambda (\epsilon_{AC} \epsilon_{BD} + \epsilon_{AD} \epsilon_{BC}), \quad \Psi_{ABCD} = \Psi_{(ABCD)} \quad (76)$$

and similarly for $\tilde{X}_{X'Y'Z'W'}$. The totally symmetric spinors Ψ_{ABCD} and its tilde counterpart $\tilde{\Psi}_{X'Y'Z'W'}$ are the Weyl spinors. By means of the relations (73)-(75) we find that the Ricci spinor is of the form

$$R_{AX'BY'} = -2\Phi_{ABX'Y'} + 6\Lambda \epsilon_{AB} \epsilon_{X'Y'} \quad (77)$$

where we note again that there is no independent tilde version of the trace-free Ricci spinor $\Phi_{ABX'Y'}$. Thus the existence of two independent spin-frames naturally leads to the definition of two independent Weyl spinors while the Ricci spinor remains unaffected. The curvature complex dyad scalars are related to the curvature spinors as follows

$$\begin{aligned} \Phi_{00} &:= \Phi_{000'0'} & \Phi_{01} &:= \Phi_{000'1'} & \Phi_{02} &:= \Phi_{001'1'} \\ \Phi_{10} &:= \Phi_{010'0'} & \Phi_{11} &:= \Phi_{010'1'} & \Phi_{12} &:= \Phi_{011'1'} \\ \Phi_{20} &:= \Phi_{110'0'} & \Phi_{21} &:= \Phi_{110'1'} & \Phi_{22} &:= \Phi_{111'1'} \end{aligned} \quad (78)$$

$$\begin{aligned} \Psi_0 &:= \Psi_{0000} & \Psi_1 &:= \Psi_{0001} & \Psi_2 &:= \Psi_{0011} \\ \Psi_3 &:= \Psi_{0111} & \Psi_4 &:= \Psi_{1111} & & \\ \tilde{\Psi}_0 &:= \tilde{\Psi}_{0'0'0'0'} & \tilde{\Psi}_1 &:= \tilde{\Psi}_{0'0'0'1'} & \tilde{\Psi}_2 &:= \tilde{\Psi}_{0'0'1'1'} \\ \tilde{\Psi}_3 &:= \tilde{\Psi}_{0'1'1'1'} & \tilde{\Psi}_4 &:= \tilde{\Psi}_{1'1'1'1'} & & \end{aligned} \quad (79)$$

and their relationship to the complex dyad components of the Riemann tensor are given by eqs.(68)-(70) which also shows that

$$\Psi_0 = \overline{\Psi}_4, \quad \Psi_1 = -\overline{\Psi}_3, \quad \tilde{\Psi}_0 = \overline{\tilde{\Psi}}_4, \quad \tilde{\Psi}_1 = -\overline{\tilde{\Psi}}_3. \quad (80)$$

Under the action of the tilde operation the Weyl and the trace-free Ricci spinors undergo the transformation

$$\begin{aligned} \Psi_0 &\leftrightarrow \tilde{\Psi}_4, & \Psi_1 &\leftrightarrow \tilde{\Psi}_3, & \Psi_2 &\leftrightarrow \tilde{\Psi}_2 \\ \Phi_{00} &\leftrightarrow \Phi_{22}, & \Phi_{01} &\leftrightarrow \Phi_{12}, & \Phi_{10} &\leftrightarrow \Phi_{21} \end{aligned} \quad (81)$$

with $\Phi_{11}, \Phi_{02}, \Phi_{20}$ unaffected.

9.1 Self-dual curvature

We shall now show that the two independent Weyl spinors Ψ_{ABCD} and $\tilde{\Psi}_{X'Y'Z'W'}$ determine the self-dual and anti-self-dual parts of curvature respectively. Beginning with the spinor equivalent of the totally anti-symmetric Levi-Civita tensor density (41) we find that the spinor analog of the dual ${}^*R_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\beta}{}^{\mu\nu} R_{\mu\nu\gamma\delta}$ is given by

$${}^*R_{AX'BY'CZ'DW'} = \frac{1}{2} \epsilon_{AX'BY'}{}^{ES'FT'} R_{ES'FT' CZ'DW'} = R_{AY'BX' CZ'DW'} \quad (82)$$

which simply flips the primed indices in the first pair. Using this property for the self-dual part of the curvature spinor (71) we find

$$\begin{aligned} {}^+R_{AX'BY'CZ'DW'} &= \frac{1}{2} (R_{AX'BY'CZ'DW'} + {}^*R_{AX'BY'CZ'DW'}) \\ &= {}^+R_{[AB](X'Y')CZ'DW'} \\ &= \tilde{\Psi}_{X'Y'Z'W'} \epsilon_{AB} \epsilon_{CD} + \Phi_{X'Y'CD} \epsilon_{AB} \epsilon_{Z'W'} \\ &\quad + \Lambda \epsilon_{AB} \epsilon_{CD} (\epsilon_{X'Z'} \epsilon_{Y'W'} + \epsilon_{X'W'} \epsilon_{Y'Z'}) \end{aligned} \quad (83)$$

while the anti-self-dual curvature spinor is given by

$$\begin{aligned} {}^-R_{AX'BY'CZ'DW'} &= \frac{1}{2} (R_{AX'BY'CZ'DW'} - {}^*R_{AX'BY'CZ'DW'}) \\ &= {}^-R_{(AB)[X'Y']CZ'DW'} \\ &= \Psi_{ABCD} \epsilon_{X'Y'} \epsilon_{Z'W'} + \Phi_{ABY'Z'} \epsilon_{X'Y'} \epsilon_{CD} \\ &\quad + \Lambda \epsilon_{X'Y'} \epsilon_{Z'W'} (\epsilon_{AC} \epsilon_{BD} + \epsilon_{AD} \epsilon_{BC}). \end{aligned} \quad (84)$$

From the above decompositions it is evident that the necessary and sufficient conditions for the spinor of curvature to be self-dual, or anti-self-dual are given by either

$$\Psi_{ABCD} \equiv 0, \quad \Phi_{ABZ'W'} \equiv 0, \quad \Lambda \equiv 0, \quad (85)$$

or

$$\tilde{\Psi}_{X'Y'Z'W'} \equiv 0, \quad \Phi_{X'Y'CD} \equiv 0, \quad \Lambda \equiv 0 \quad (86)$$

which illustrates the well known fact that (anti)-self-duality of the Riemann curvature tensor implies Ricci-flatness and is a much stronger requirement than Ricci-flatness itself.

10 Curvature 2-forms

For gravitational instantons curvature is an \mathcal{SO}_4 -valued 2-form. The spinor equivalent of the Riemann curvature 2-form splits naturally into its self-dual and anti-self-dual parts

$$\begin{aligned}\theta_{\alpha\beta} \leftrightarrow \theta_{AX'BY'} &= \frac{1}{2} R_{AX'BY' CZ'DW'} \sigma^{CZ'} \wedge \sigma^{DW'} \\ &= \Theta_{AB} \epsilon_{X'Y'} + \tilde{\Theta}_{X'Y'} \epsilon_{AB}\end{aligned}\quad (87)$$

that is, we have the two independent symmetric matrices of curvature 2-forms

$$\begin{aligned}\Theta_A^B &= \frac{1}{4} R_{AX' CZ'DW'}^{BX'} \sigma^{CZ'} \wedge \sigma^{DW'} \\ \tilde{\Theta}_{X'}^{Y'} &= \frac{1}{4} R_{AX' CZ'DW'}^{AY'} \sigma^{CZ'} \wedge \sigma^{DW'}.\end{aligned}\quad (88)$$

Using the equation (71) together with eqs.(50) we find

$$\begin{aligned}\Theta_{AB} +^* \Theta_{AB} &= 2 \Phi_{ABX'Y'} L^{X'Y'} \\ \Theta_{AB} -^* \Theta_{AB} &= 2 [\Psi_{ABCD} + \Lambda (\epsilon_{AC} \epsilon_{BD} + \epsilon_{AD} \epsilon_{BC})] L^{CD}\end{aligned}\quad (89)$$

so that the components of the spinor valued curvature 2-form Θ_A^B in terms of the various complex dyad scalars are given by

$$\begin{aligned}\Theta_0^0 &= -(\Psi_2 - \Lambda + \Phi_{11}) l \wedge \bar{l} - (\Psi_2 - \Lambda - \Phi_{11}) m \wedge \bar{m} \\ &\quad - \bar{\Psi}_1 l \wedge m + \Psi_1 \bar{l} \wedge \bar{m} - \Phi_{10} \bar{l} \wedge m - \Phi_{12} l \wedge \bar{m}, \\ \Theta_0^1 &= (\Psi_1 + \Phi_{01}) l \wedge \bar{l} + (\Psi_1 - \Phi_{01}) m \wedge \bar{m} + \Phi_{02} l \wedge \bar{m} \\ &\quad - (\Psi_2 + 2\Lambda) l \wedge m - \Psi_0 \bar{l} \wedge \bar{m} + \Phi_{00} \bar{l} \wedge m, \\ \Theta_1^0 &= (\bar{\Psi}_1 - \Phi_{21}) l \wedge \bar{l} + (\bar{\Psi}_1 + \Phi_{21}) m \wedge \bar{m} - \Phi_{22} l \wedge \bar{m} \\ &\quad + \bar{\Psi}_0 l \wedge m + (\Psi_2 + 2\Lambda) \bar{l} \wedge \bar{m} - \Phi_{20} \bar{l} \wedge m = -\bar{\Theta}_0^1\end{aligned}\quad (90)$$

and similarly

$$\begin{aligned}
\tilde{\Theta}_{0'}^{0'} &= -(\tilde{\Psi}_2 - \Lambda + \Phi_{11})l \wedge \bar{l} + (\tilde{\Psi}_2 - \Lambda - \Phi_{11})m \wedge \bar{m} \\
&\quad - \tilde{\Psi}_1 \bar{l} \wedge m + \overline{\tilde{\Psi}}_1 l \wedge \bar{m} + \Phi_{01} \bar{l} \wedge \bar{m} + \Phi_{21} l \wedge m, \\
\tilde{\Theta}_{0'}^{1'} &= (\tilde{\Psi}_1 + \Phi_{10})l \wedge \bar{l} - (\tilde{\Psi}_1 - \Phi_{10})m \wedge \bar{m} - \Phi_{20} l \wedge m \\
&\quad + (\tilde{\Psi}_2 + 2\Lambda)l \wedge \bar{m} + \tilde{\Psi}_0 \bar{l} \wedge m - \Phi_{00} \bar{l} \wedge \bar{m}, \\
\tilde{\Theta}_{1'}^{0'} &= (\overline{\tilde{\Psi}}_1 - \Phi_{12})l \wedge \bar{l} - (\overline{\tilde{\Psi}}_1 + \Phi_{12})m \wedge \bar{m} + \Phi_{22} l \wedge m \\
&\quad - \overline{\tilde{\Psi}}_0 l \wedge \bar{m} - (\tilde{\Psi}_2 + 2\Lambda)\bar{l} \wedge m + \Phi_{02} \bar{l} \wedge \bar{m} = -\overline{\tilde{\Theta}_{0'}}^{1'}
\end{aligned} \tag{91}$$

where we note that under the tilde operation eqs.(90) and (91) go into each other according to the rules (25), (81).

11 Ricci identities

The curvature 2-forms (90) are related to the connection 1-forms (36) through

$$\Theta_a^b = d\Gamma_a^b - \Gamma_a^c \wedge \Gamma_c^b \tag{92}$$

and of course the same relation holds for the tilde quantities as well. The full set of Ricci identities is the explicit form of these definitions of curvature in terms of the spin coefficients and curvature complex dyad scalars. We find

that eqs.(92) result in the Ricci identities

$$\begin{aligned}
D\sigma - \delta\kappa + \kappa(\tau + \beta + 3\bar{\alpha} - \pi) - \sigma(\rho + 3\epsilon + \bar{\gamma} - \bar{\lambda}) &= \Psi_0 \\
D\bar{\alpha} - \delta\epsilon - \kappa(\bar{\epsilon} + \bar{\rho}) - \bar{\alpha}(\bar{\gamma} - \bar{\lambda}) - \sigma(\alpha + \bar{\tau}) - \epsilon(\pi - \beta) &= \Psi_1 \\
\delta\alpha + \bar{\delta}\bar{\alpha} + \epsilon(\lambda + \bar{\rho}) + \bar{\epsilon}(\bar{\lambda} + \rho) + \alpha(\bar{\alpha} - \beta) + \bar{\alpha}(\alpha - \bar{\beta}) + \rho\bar{\rho} - \sigma\bar{\sigma} \\
&= -\Psi_2 + \Phi_{11} + \Lambda \\
D\bar{\epsilon} + \bar{D}\epsilon - \epsilon(\gamma - \bar{\epsilon}) - \bar{\epsilon}(\bar{\gamma} - \epsilon) + \alpha(\pi + \tau) + \bar{\alpha}(\bar{\pi} + \bar{\tau}) + \tau\bar{\tau} - \kappa\bar{\kappa} \\
&= -\Psi_2 - \Phi_{11} + \Lambda \\
D\bar{\rho} + \delta\bar{\tau} - \bar{\tau}(\beta - \pi - \bar{\alpha}) - \bar{\rho}(\bar{\gamma} - \bar{\lambda} - \epsilon) - \kappa\bar{\kappa} - \sigma\bar{\sigma} &= -\Psi_2 - 2\Lambda \\
D\tau - \bar{D}\kappa - \tau(\bar{\gamma} + \epsilon) + \kappa(\gamma - 3\bar{\epsilon}) - \rho(\pi + \tau) - \sigma(\bar{\pi} + \bar{\tau}) &= \Psi_1 + \Phi_{01} \\
\delta\rho + \bar{\delta}\sigma + \sigma(3\alpha - \bar{\beta}) + \kappa(\lambda + \bar{\rho}) - \rho(\bar{\alpha} + \beta) - \tau(\rho + \bar{\lambda}) &= \Phi_{01} - \Psi_1 \\
\bar{D}\bar{\alpha} + \delta\bar{\epsilon} - \bar{\epsilon}(\tau - \beta - \bar{\alpha}) + \bar{\alpha}(\bar{\epsilon} - \bar{\rho} + \gamma) + \alpha\mu - \epsilon\nu - \tau\bar{\rho} - \sigma\bar{\kappa} &= -\Phi_{12} \\
D\rho + \bar{\delta}\kappa + \rho(\bar{\gamma} - \rho - \epsilon) - \kappa(\bar{\tau} - \bar{\beta} - 3\alpha) + \tau\bar{\nu} + \sigma\bar{\mu} &= \Phi_{00} \\
\delta\tau - \bar{D}\sigma - \tau(\tau - \beta + \bar{\alpha}) - \sigma(3\bar{\epsilon} - \bar{\rho} + \gamma) - \rho\mu + \kappa\nu &= \Phi_{02}
\end{aligned} \tag{93}$$

and the remaining set which follows from the relation

$$\tilde{\Theta}_{x'}^{y'} = d\tilde{\Gamma}_{x'}^{y'} - \tilde{\Gamma}_{x'}^{z'} \wedge \tilde{\Gamma}_{z'}^{y'} \tag{94}$$

is given by

$$\begin{aligned}
\bar{\delta}\bar{\nu} - D\bar{\mu} - \bar{\nu}(\bar{\tau} - \bar{\pi} - 3\bar{\beta} - \alpha) - \bar{\mu}(\epsilon - \rho + 3\bar{\gamma} + \bar{\lambda}) &= \tilde{\Psi}_0 \\
D\bar{\beta} - \bar{\delta}\bar{\gamma} + \bar{\beta}(\epsilon - \rho) + \bar{\gamma}(\bar{\tau} - \alpha) + \bar{\nu}(\lambda + \gamma) + \bar{\mu}(\pi + \beta) &= \tilde{\Psi}_1 \\
D\gamma + \bar{D}\bar{\gamma} - \gamma(\bar{\gamma} - \epsilon) - \bar{\gamma}(\gamma - \bar{\epsilon}) - \bar{\beta}(\pi + \tau) - \beta(\bar{\pi} + \bar{\tau}) - \pi\bar{\pi} + \nu\bar{\nu} \\
&= \tilde{\Psi}_2 + \Phi_{11} - \Lambda \\
\delta\bar{\beta} + \bar{\delta}\beta + \beta(\alpha - \bar{\beta}) + \bar{\beta}(\bar{\alpha} - \beta) - \gamma(\bar{\lambda} + \rho) - \bar{\gamma}(\lambda + \bar{\rho}) - \lambda\bar{\lambda} + \mu\bar{\mu} \\
&= \tilde{\Psi}_2 - \Phi_{11} - \Lambda \\
\delta\bar{\pi} + \bar{D}\bar{\lambda} - \bar{\pi}(\tau + \beta - \bar{\alpha}) + \bar{\lambda}(\bar{\epsilon} - \bar{\rho} - \gamma) + \mu\bar{\mu} + \nu\bar{\nu} &= \tilde{\Psi}_2 + 2\Lambda \\
D\bar{\pi} - \bar{D}\bar{\nu} + \bar{\pi}(\bar{\gamma} + \epsilon) + \bar{\nu}(3\gamma - \bar{\epsilon}) + \bar{\mu}(\pi + \tau) + \bar{\lambda}(\bar{\pi} + \bar{\tau}) &= \tilde{\Psi}_1 + \Phi_{10} \\
\delta\bar{\mu} + \bar{\delta}\bar{\lambda} + \bar{\mu}(\bar{\alpha} - 3\beta) + \bar{\lambda}(\alpha + \bar{\beta}) + \bar{\pi}(\bar{\lambda} + \rho) - \bar{\nu}(\lambda + \bar{\rho}) &= -\tilde{\Psi}_1 + \Phi_{10} \\
\delta\bar{\gamma} + D\beta - \bar{\gamma}(\bar{\alpha} + \beta - \pi) - \beta(\epsilon + \bar{\gamma} - \bar{\lambda}) - \sigma\bar{\beta} + \pi\bar{\lambda} + \mu\bar{\nu} + \gamma\kappa &= \Phi_{01} \\
D\bar{\lambda} + \delta\bar{\nu} - \bar{\lambda}(\epsilon - \bar{\gamma} - \bar{\lambda}) - \bar{\nu}(\bar{\alpha} + 3\beta - \pi) - \bar{\mu}\sigma - \kappa\bar{\pi} &= \Phi_{00} \\
D\mu - \delta\pi - \mu(\epsilon + 3\bar{\gamma} - \bar{\lambda}) + \pi(\bar{\alpha} - \beta - \pi) + \kappa\nu - \lambda\sigma &= \Phi_{02}.
\end{aligned} \tag{95}$$

We note that the last two equations in each set are equivalent, thus eqs.(93) and (95) together with their complex conjugates consist of a full set of Ricci identities. Furthermore these identities are related to each other by the tilde operation according to eqs.(26), (81).

12 Bianchi identities

Bianchi had noted [23] two sets of identities satisfied by the Riemann tensor. The cyclic identity

$$e^b \wedge \theta_b^a = 0 \tag{96}$$

is the first one of Bianchi's identities. At this point it is worth noting that, quite generally, the left hand side of Einstein field equations are given by the 3-form

$$e^b \wedge {}^*\theta_b^a = 0 \tag{97}$$

so that when the curvature 2-form is self-dual, it is the first Bianchi identity that assures Ricci-flatness. This is unlike the case of Maxwell field where

one half of Maxwell equations which is the analog of second Bianchi identities implies the satisfaction of source-free Maxwell equations for a self-dual Maxwell 2-form.

The second Bianchi identities split into two independent sets which are given by

$$d\Theta_a{}^b + \Theta_a{}^c \wedge \Gamma_c{}^b - \Gamma_a{}^c \wedge \Theta_c{}^b = 0 \quad (98)$$

together with its tilde version corresponding to the two spin frames. Inserting the expressions (90) and (36) into eq.(98) we find the first set of second Bianchi identities

$$\begin{aligned} & (\bar{D} + 2\bar{\epsilon} - 2\bar{\rho}) \Psi_1 - (\delta - 3\tau) \Psi_2 + 2\sigma\bar{\Psi}_1 - \bar{\kappa} \Psi_0 + (D - 2\bar{\gamma} + \bar{\lambda}) \Phi_{12} \\ & \quad - (\delta + 2\pi) \Phi_{11} + \delta\Lambda + \mu\Phi_{10} + \kappa\Phi_{22} - \sigma\Phi_{21} - \bar{\tau} \Phi_{02} - \bar{\rho} \Phi_{01} = 0 \\ & (\bar{D} + 2\bar{\epsilon} - 2\bar{\rho}) \Psi_1 - (\delta - 3\tau) \Psi_2 + 2\sigma\bar{\Psi}_1 - \bar{\kappa} \Psi_0 - (\bar{D} + 2\bar{\epsilon} + 2\lambda) \Phi_{01} \\ & \quad - (\bar{\delta} + \bar{\pi} + 2\alpha - 2\bar{\beta}) \Phi_{02} - 2\delta\Lambda - 2\tau\Phi_{11} + \nu\Phi_{00} + 2\rho\Phi_{12} = 0 \\ & (\delta - 4\tau - 2\bar{\alpha}) \Psi_1 - (\bar{D} + 4\bar{\epsilon} - \bar{\rho}) \Psi_0 + 3\sigma\Psi_2 + (\delta + 2\pi - 2\bar{\alpha}) \Phi_{01} \\ & \quad - (D - 2\epsilon - 2\bar{\gamma} + \bar{\lambda}) \Phi_{02} - \mu\Phi_{00} - 2\kappa\Phi_{12} + 2\sigma\Phi_{11} = 0 \\ & (D - 4\rho - 2\epsilon) \Psi_1 + (\bar{\delta} + 4\alpha - \bar{\tau}) \Psi_0 + 3\kappa\Psi_2 + (\delta + \pi - 2\bar{\alpha} - 2\beta) \Phi_{00} \\ & \quad - (D - 2\epsilon + 2\bar{\lambda}) \Phi_{01} - 2\kappa\Phi_{11} + 2\sigma\Phi_{10} - \bar{\nu} \Phi_{02} = 0 \\ & (\bar{\delta} + 2\alpha - 2\bar{\tau}) \Psi_1 + (D - 3\rho) \Psi_2 - 2\kappa\bar{\Psi}_1 - \bar{\sigma} \Psi_0 + (\delta + \pi - 2\beta) \Phi_{10} \\ & \quad - (D + 2\bar{\lambda}) \Phi_{11} - D\Lambda - \kappa\Phi_{21} + \sigma\Phi_{20} - \bar{\nu} \Phi_{12} + \bar{\tau} \Phi_{01} + \bar{\rho}\Phi_{00} = 0 \\ & (\bar{\delta} + 2\alpha - 2\bar{\tau}) \Psi_1 + (D - 3\rho) \Psi_2 - 2\kappa\bar{\Psi}_1 - \bar{\sigma} \Psi_0 + (\bar{\delta} + 2\bar{\pi} + 2\alpha) \Phi_{01} \\ & \quad + (\bar{D} + \lambda + 2\bar{\epsilon} - 2\gamma) \Phi_{00} + 2D\Lambda - 2\rho\Phi_{11} + \bar{\mu} \Phi_{02} + 2\tau\Phi_{10} = 0 \end{aligned} \quad (99)$$

The remaining set of Bianchi identities is the tilde version of eqs.(98). In

terms of the spin coefficients and complex dyad scalars they are given by

$$\begin{aligned}
& (\bar{D} + 2\lambda - 2\gamma) \tilde{\Psi}_1 + (\bar{\delta} + 3\bar{\pi}) \tilde{\Psi}_2 - 2\bar{\mu} \tilde{\bar{\Psi}}_1 - \nu \tilde{\Psi}_0 + (D + 2\epsilon - \rho) \Phi_{21} \\
& \quad + (\bar{\delta} - 2\bar{\tau}) \Phi_{11} - \bar{\delta} \Lambda - \pi \Phi_{20} + \lambda \Phi_{10} - \bar{\sigma} \Phi_{01} + \bar{\mu} \Phi_{12} + \bar{\nu} \Phi_{22} = 0 \\
& (\bar{D} + 2\lambda - 2\gamma) \tilde{\Psi}_1 + (\bar{\delta} + 3\bar{\pi}) \tilde{\Psi}_2 - 2\bar{\mu} \tilde{\bar{\Psi}}_1 - \nu \tilde{\Psi}_0 - (\bar{D} - 2\gamma - 2\bar{\rho}) \Phi_{10} \\
& \quad + (\delta + 2\bar{\alpha} - 2\beta - \tau) \Phi_{20} + 2\bar{\delta} \Lambda + \bar{\kappa} \Phi_{00} - 2\bar{\pi} \Phi_{11} - 2\bar{\lambda} \Phi_{21} = 0 \\
& (\bar{\delta} + 4\bar{\pi} + 2\bar{\beta}) \tilde{\Psi}_1 + (\bar{D} - 4\gamma + \lambda) \tilde{\Psi}_0 + 3\bar{\mu} \tilde{\Psi}_2 + (\bar{\delta} + 2\bar{\beta} - 2\bar{\tau}) \Phi_{10} \\
& \quad + (D + 2\epsilon + 2\bar{\gamma} - \rho) \Phi_{20} - \bar{\sigma} \Phi_{00} + 2\bar{\nu} \Phi_{21} + 2\bar{\mu} \Phi_{11} = 0 \\
& (D + 4\bar{\lambda} + 2\bar{\gamma}) \tilde{\Psi}_1 - (\delta - 4\beta + \pi) \tilde{\Psi}_0 + 3\bar{\nu} \tilde{\Psi}_2 - (D - 2\rho + 2\bar{\gamma}) \Phi_{10} \\
& \quad - (\bar{\delta} + 2\bar{\beta} + 2\alpha - \bar{\tau}) \Phi_{00} - \kappa \Phi_{20} - 2\bar{\nu} \Phi_{11} - 2\bar{\mu} \Phi_{01} = 0 \\
& (\delta + 2\pi - 2\beta) \tilde{\Psi}_1 - (D + 3\bar{\lambda}) \tilde{\Psi}_2 + 2\bar{\nu} \tilde{\bar{\Psi}}_1 - \mu \tilde{\Psi}_0 + (\bar{\delta} - \bar{\tau} + 2\alpha) \Phi_{01} \\
& \quad + (D - 2\rho) \Phi_{11} + D\Lambda + \kappa \Phi_{21} + \lambda \Phi_{00} - \pi \Phi_{10} + \bar{\nu} \Phi_{12} + \bar{\mu} \Phi_{02} = 0 \\
& (\delta + 2\pi - 2\beta) \tilde{\Psi}_1 - (D + 3\bar{\lambda}) \tilde{\Psi}_2 + 2\bar{\nu} \tilde{\bar{\Psi}}_1 - \mu \tilde{\Psi}_0 + (\delta - 2\beta - 2\tau) \Phi_{10} \\
& \quad - (\bar{D} + 2\bar{\epsilon} - 2\gamma - \bar{\rho}) \Phi_{00} - 2D\Lambda + \sigma \Phi_{20} - 2\bar{\pi} \Phi_{01} - 2\bar{\lambda} \Phi_{11} = 0
\end{aligned} \tag{100}$$

Looking at the first and last pair of equations in both sets of eqs.(99) and (100) we note that in each pair one equation is equivalent to the other, so that we have only the eight identities which together with their complex conjugates provide us with the full set of the Bianchi identities. Furthermore, from the first and last pair of equations in (99) and (100) we obtain the contracted Bianchi identities involving only the Ricci scalars

$$\begin{aligned}
& (D - 2\bar{\gamma} + \bar{\lambda}) \Phi_{12} - (\delta + 2\pi) \Phi_{11} + \delta \Lambda + \mu \Phi_{10} \\
& \quad + \kappa \Phi_{22} - \sigma \Phi_{21} - \bar{\tau} \Phi_{02} - \bar{\rho} \Phi_{01} \\
& = -(\bar{D} + 2\bar{\epsilon} + 2\lambda) \Phi_{01} - (\bar{\delta} + \bar{\pi} + 2\alpha - 2\bar{\beta}) \Phi_{02} \\
& \quad - 2\delta \Lambda - 2\tau \Phi_{11} + \nu \Phi_{00} + 2\rho \Phi_{12} \\
& (\delta + \pi - 2\beta) \Phi_{10} - (D + 2\bar{\lambda}) \Phi_{11} - D\Lambda - \kappa \Phi_{21} \\
& \quad + \sigma \Phi_{20} - \bar{\nu} \Phi_{12} + \bar{\tau} \Phi_{01} + \bar{\rho} \Phi_{00} \\
& = (\bar{\delta} + 2\bar{\pi} + 2\alpha) \Phi_{01} + (\bar{D} + \lambda + 2\bar{\epsilon} - 2\gamma) \Phi_{00} \\
& \quad + 2D\Lambda - 2\rho \Phi_{11} + \bar{\mu} \Phi_{02} + 2\tau \Phi_{10}
\end{aligned} \tag{101}$$

We note that the tilde operation swaps both sets of the above Bianchi identities according to eqs.(26), (81).

13 Maxwell field

The discussion of the (anti)-self-dual properties of the electromagnetic field in 4-dimensional Riemannian manifold with positive definite signature follows closely the general scheme we have presented for the Ricci and second Bianchi identities. In particular, the source-free Maxwell equations also fall into two independent sets. To show this we start by the decomposition of the Maxwell 2-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} F_{AX'BY'} \sigma^{AX'} \wedge \sigma^{BY'} \quad (102)$$

into its self-dual and anti-self-dual parts using, once again, the relation

$$F_{AX'BY'} = \varphi_{AB} \epsilon_{X'Y'} + \tilde{\varphi}_{X'Y'} \epsilon_{AB} \quad (103)$$

where φ_{AB} and $\tilde{\varphi}_{X'Y'}$ are symmetric spinors, together with (48). Explicitly we find that the (anti)-self-dual Maxwell 2-forms are given by

$$\begin{aligned} \mathcal{F} &= \frac{1}{2} (F - *F) = \varphi_{AB} L^{AB} \\ &= -\varphi_1 (l \wedge \bar{l} + m \wedge \bar{m}) + \varphi_0 \bar{l} \wedge \bar{m} + \bar{\varphi}_0 l \wedge m \\ \tilde{\mathcal{F}} &= \frac{1}{2} (F + *F) = \tilde{\varphi}_{X'Y'} \tilde{L}^{X'Y'} \\ &= -\tilde{\varphi}_1 (l \wedge \bar{l} - m \wedge \bar{m}) - \tilde{\varphi}_0 \bar{l} \wedge m - \bar{\tilde{\varphi}}_0 l \wedge \bar{m} \end{aligned} \quad (104)$$

where we have introduced the complex dyad scalars of the Maxwell field

$$\begin{aligned} \varphi_1 &:= \varphi_{01} = \frac{1}{2} F_{\mu\nu} (l^\mu \bar{l}^\nu + m^\mu \bar{m}^\nu), & \varphi_0 &:= \varphi_{00} = F_{\mu\nu} l^\mu m^\nu \\ \tilde{\varphi}_1 &:= \tilde{\varphi}_{0'1'} = \frac{1}{2} F_{\mu\nu} (l^\mu \bar{l}^\nu - m^\mu \bar{m}^\nu), & \tilde{\varphi}_0 &:= \tilde{\varphi}_{0'0'} = F_{\mu\nu} \bar{m}^\mu l^\nu \\ \bar{\varphi}_0 &:= \varphi_{11}, & \bar{\tilde{\varphi}}_0 &:= \tilde{\varphi}_{1'1'} \end{aligned} \quad (105)$$

which add up to the six real components of the Maxwell field. Under the tilde operation the complex dyad scalars manifest the correspondence

$$\varphi_1 \xleftrightarrow{\sim} -\tilde{\varphi}_1, \quad \varphi_0 \xleftrightarrow{\sim} -\bar{\tilde{\varphi}}_0 \quad (106)$$

We conclude that in the spinor formalism for the positive definite signature the electromagnetic fields are completely described by means of two complex (φ_0 , $\tilde{\varphi}_0$) and two purely imaginary (φ_1 , $\tilde{\varphi}_1$) complex dyad scalars.

13.1 Source-free Maxwell equations

In vacuum the self-dual and anti-self-dual 2-forms (104) both satisfy the source-free Maxwell equations

$$d\mathcal{F} = 0, \quad d\tilde{\mathcal{F}} = 0, \quad (107)$$

and from eqs.(104) and (107) using the relations (8) and (21) we arrive at the two independent sets of the equations

$$\begin{aligned} (\delta - 2\tau) \varphi_1 - (\bar{D} + 2\bar{\epsilon} - \bar{\rho}) \varphi_0 + \sigma \bar{\varphi}_0 &= 0 \\ (D - 2\rho) \varphi_1 + (\bar{\delta} + 2\alpha - \bar{\tau}) \varphi_0 + \kappa \bar{\varphi}_0 &= 0 \end{aligned} \quad (108)$$

and

$$\begin{aligned} (\bar{\delta} + 2\bar{\pi}) \tilde{\varphi}_1 + (\bar{D} - 2\gamma + \lambda) \tilde{\varphi}_0 + \bar{\mu} \bar{\tilde{\varphi}}_0 &= 0 \\ (D + 2\bar{\lambda}) \tilde{\varphi}_1 - (\delta - 2\beta + \pi) \tilde{\varphi}_0 + \bar{\nu} \bar{\tilde{\varphi}}_0 &= 0 \end{aligned} \quad (109)$$

which consist of the Newman-Penrose version of the full set of source-free Maxwell equations in Euclidean signature. It is clear that the equations (108) and (109) go into each other under the tilde operation according to the correspondences (23), (26) and (106).

14 Topological numbers

The Newman-Penrose formalism for Euclidean signature simplifies the discussion of the topological invariants of gravitational instantons. We shall express the Euler number and Hirzebruch signature as integrals over the spinor equivalents of curvature and connection on a manifold \mathcal{M} whereby the corresponding self-dual and anti-self-dual contributions become explicitly manifest. We begin with the Chern formulae [24], [25] for the Euler number

$$\begin{aligned} \chi &= \frac{1}{32\pi^2} \int_{\mathcal{M}} \epsilon^{abcd} \theta_{ab} \wedge \theta_{cd} \\ &\quad - \frac{1}{16\pi^2} \int_{\partial\mathcal{M}} \epsilon^{abcd} \left(\Omega_{ab} \wedge \theta_{cd} - \frac{2}{3} \Omega_{ab} \wedge \Omega_c^e \wedge \Omega_{ed} \right) \end{aligned} \quad (110)$$

and the Hirzebruch signature

$$\tau = -\frac{1}{24\pi^2} \int_{\mathcal{M}} \theta_a^b \wedge \theta_b^a + \frac{1}{24\pi^2} \int_{\partial\mathcal{M}} \Omega_a^b \wedge \theta_b^a - \eta_s(\partial\mathcal{M}) \quad (111)$$

of a non-compact manifold. Here θ_{ab} is the curvature 2-form, Ω_{ab} is the second fundamental form of the boundary $\partial\mathcal{M}$

$$\Omega_{ab} = \omega_{ab} - (\omega_0)_{ab} \quad (112)$$

which is the difference between the connection 1-form ω_{ab} for the original metric and the connection 1-form $(\omega_0)_{ab}$ of a product (induced) metric on the boundary $\partial\mathcal{M}$. Finally $\eta_s(\partial\mathcal{M})$ is the eta-invariant [26], [27].

Using the spinor equivalent of the totally anti-symmetric Levi-Civita alternating symbol (41), the connection 1-form (42), and the curvature 2-form (87) we obtain the spinor expression for the Euler number

$$\begin{aligned} \chi = & \frac{1}{8\pi^2} \int_{\mathcal{M}} \left(\Theta_A^B \wedge \Theta_B^A - \tilde{\Theta}_{A'}^{B'} \wedge \tilde{\Theta}_{B'}^{A'} \right) \\ & - \frac{1}{4\pi^2} \int_{\partial\mathcal{M}} \left[\gamma_A^B \wedge \Theta_B^A - \tilde{\gamma}_{A'}^{B'} \wedge \tilde{\Theta}_{B'}^{A'} \right. \\ & \left. - \frac{2}{3} \left(\gamma_A^B \wedge \gamma_B^E \wedge \gamma_E^A - \tilde{\gamma}_{A'}^{B'} \wedge \tilde{\gamma}_{B'}^{E'} \wedge \tilde{\gamma}_{E'}^{A'} \right) \right] \quad (113) \end{aligned}$$

where

$$\gamma_A^B = \Gamma_A^B - (\Gamma_0)_A^B \quad (114)$$

is the spinor equivalent of the second fundamental form. We see that the self-dual and anti-self-dual contributions to this integral are explicit.

In the case of a compact manifold when the boundary terms vanish the spinor expression of the Euler number (113) can be reduced into the form

$$\begin{aligned} \chi = & \frac{1}{4\pi^2} \int_{\mathcal{M}} \left[|\Psi_0|^2 + 4|\Psi_1|^2 + 3\Psi_2^2 + |\tilde{\Psi}_0|^2 \right. \\ & + 4|\tilde{\Psi}_1|^2 + 3\tilde{\Psi}_2^2 - 2(|\Phi_{00}|^2 + |\Phi_{02}|^2) \\ & \left. - 4(|\Phi_{01}|^2 + |\Phi_{11}|^2 + |\Phi_{12}|^2 - 3\Lambda^2) \right] l \wedge \bar{l} \wedge m \wedge \bar{m} \quad (115) \end{aligned}$$

involving the Weyl and the Ricci scalars.

Similarly, the Hirzebruch signature can be expressed as integrals of the spinor-valued curvature 2-forms and connection 1-forms. We find the expression

$$\tau = -\frac{1}{12\pi^2} \int_{\mathcal{M}} \left(\Theta_A^B \wedge \Theta_B^A + \tilde{\Theta}_{A'}^{B'} \wedge \tilde{\Theta}_{B'}^{A'} \right)$$

$$- \int_{\partial \mathcal{M}} (\gamma_A^B \wedge \Theta_B^A + \tilde{\gamma}_{A'}^{B'} \wedge \tilde{\Theta}_{B'}^{A'}) \Big] - \eta_s(\partial \mathcal{M}). \quad (116)$$

which, in turn, goes to the form

$$\begin{aligned} \tau = & -\frac{1}{6\pi^2} \int_{\mathcal{M}} \left[|\Psi_0|^2 + 4|\Psi_1|^2 + 3\Psi_2^2 - |\tilde{\Psi}_0|^2 \right. \\ & \left. - 4|\tilde{\Psi}_1|^2 - 3\tilde{\Psi}_2^2 \right] l \wedge \bar{l} \wedge m \wedge \bar{m} - \eta_s(\partial \mathcal{M}) \end{aligned} \quad (117)$$

for a compact manifold.

15 Petrov types

We shall now discuss Petrov type classification of gravitational instantons in the spinor formalism for positive definite signature in order to point out that the basis bi-vectors used in the classification of the Riemann tensor are obtained from the almost complex structure vector-valued 1-forms in hyper-Kähler structure by lowering the vector index. For Petrov classification we examine the eigenspinors and the corresponding eigenvalues of the totally symmetric Weyl spinors (76). Following to the general scheme [9] we begin with two sets of orthonormal triads of spinors that correspond to two independent spin frame with bases (9) and (10) respectively. These triads are simply the complex structure bi-spinors (62)

$$\begin{aligned} n_{AB}^1 &= -\frac{i}{\sqrt{2}} (o_A o_B - \iota_A \iota_B) \\ n_{AB}^2 &= \frac{1}{\sqrt{2}} (o_A o_B + \iota_A \iota_B) \\ n_{AB}^3 &= i\sqrt{2} o_{(A} \iota_{B)} \end{aligned} \quad (118)$$

where it is clear that this triad satisfies the orthonormality condition

$$n_{AB}^i n_j^{AB} = \delta_j^i \quad i, j = 1, 2, 3 \quad (119)$$

and similar relations hold for primed indices. Projecting the totally symmetric Weyl spinor Ψ_{ABCD} onto the triad basis (118) we obtain the following

traceless 3×3 matrix

$$\Psi = \begin{pmatrix} \Psi_2 - \frac{1}{2}(\Psi_0 + \bar{\Psi}_0) & -\frac{i}{2}(\Psi_0 - \bar{\Psi}_0) & \Psi_1 + \bar{\Psi}_1 \\ -\frac{i}{2}(\Psi_0 - \bar{\Psi}_0) & \Psi_2 + \frac{1}{2}(\Psi_0 + \bar{\Psi}_0) & i(\Psi_1 - \bar{\Psi}_1) \\ \Psi_1 + \bar{\Psi}_1 & i(\Psi_1 - \bar{\Psi}_1) & -2\Psi_2 \end{pmatrix} \quad (120)$$

where we have used the relations (80). Since this matrix is real and symmetric it can always be diagonalised in the orthonormal basis with eigenspinors (118). Corresponding to three distinct real eigenvalues

$$\lambda_1 = \Psi_2 - \Psi_0, \quad \lambda_2 = \Psi_2 + \Psi_0, \quad \lambda_3 = -2\Psi_2 \quad (121)$$

obeying the condition

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

we have the case of algebraically general, or Petrov type I , gravitational instantons. When two roots coincide $\lambda_1 = \lambda_2$, from eqs.(121) and (80) we obtain

$$\Psi_2 = \lambda_1 = \lambda_2, \quad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \quad (122)$$

which corresponds to the algebraically special, or Petrov type D , gravitational instantons. For the anti-self-dual case a similar analysis holds for the independent primed 3×3 matrix. These results are in agreement with those obtained in [28], [7]. Thus gravitational instantons with (anti)-self-dual curvature must be either algebraically general of Petrov type I , or algebraically special of Petrov type D . We have seen that the basis bi-vectors used in the Petrov classification of the Weyl tensor are directly related to hyper-Kähler structure.

16 Conclusion

We have presented the NP formalism in Euclidean signature in terms of differential forms in the complex isotropic dyad basis and its spinor equivalent. This formalism is naturally adopted to the discussion of the physical and mathematical properties of gravitational instantons where the curvature is self-dual. As part of a systematic exposition of this formalism we have presented some known results, however, we feel that this NP formalism can be

used in many problems of current interest in both physics and mathematics. In particular, we have presented the explicit expression for the vector-valued 1-forms that define three almost complex structures for a general gravitational instanton metric which admits a self-dual curvature 2-form. We have found that the integrability condition for these almost complex structures, namely the vanishing of the Nijenhuis vector-valued 2-form is automatically satisfied in the self-dual gauge which is guaranteed by the condition of self-dual curvature. This is explicit proof of hyper-Kähler structure. The essential tool which makes this general result possible is the Newman-Penrose formalism for Euclidean signature. We have shown that the Ricci and the Bianchi identities, as well as the Maxwell equations, naturally fall into two independent sets which express their self-dual and anti-self-dual content. Finally we have shown the relationship between hyper-Kähler structure for gravitational instantons and Petrov classification.

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